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Waves Against a 135° Dock

We apply the method of Lewy* and Stoker to find progressing waves against a 135° dock.

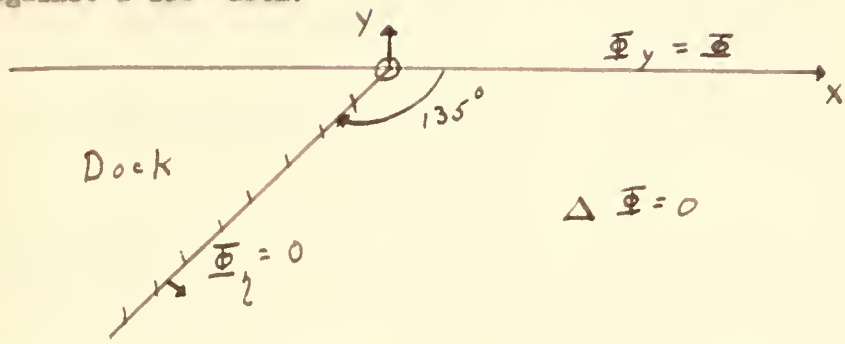

Fig 1

We seek two potential functions Φ_0 and Φ_1 which satisfy the boundary conditions:

1) $\Phi_y - \Phi = 0$ on $y = 0,\ x > 0$

2) $\Phi_\eta = 0$ on the dock $y = x,\ y < 0$ ($\frac{\partial}{\partial \eta} \equiv$ normal derivativ

In addition, Φ_0 and Φ_1 should behave at ∞ like $\mathrm{Re}(e^{i\alpha_1} e^{-iz})$ and $\mathrm{Re}(e^{i(\alpha_1 + \frac{\pi}{2})} e^{-iz})$.

We permit Φ_1 to have a logarithmic singularity at the origin, but we require that Φ_{00} be finite at the origin.

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* For angles $\frac{p\pi}{2q}$, our case corresponds to $p = 3,\ q = 2$.

The boundary conditions may be formulated for the analytic functions $F_0(z)$ and $F_1(z)$ which correspond to Φ_0 and Φ_1:

3) $\quad \mathrm{Re}(iD-1)F = 0 \qquad \text{for} \qquad z = x,\ x > 0$

4) $\quad \mathrm{Re}(ie^{-i\pi 3/4}D)F = 0 \qquad \text{for} \qquad z = re^{-i\pi 3/4},\ r > 0.$

The functions F_0 and F_1 are obtained from suitable combinations of solutions of the following linear ordinary differential equation

5) $\quad (D-1)\,(iD-1)f_0(z) = 0 \quad , \qquad\qquad z = x + iy$

6) $\quad (D-1)\,(iD-1)f_1(z) = A\,\dfrac{1}{z^{2/3}} \quad , \qquad A = \text{real constant}$

7) $\quad (D-1)\,(iD-1)f_{-1}(z) = \dfrac{B1}{z^2} \quad , \qquad B = \text{real constant}$

The right hand sides have the property that they are pure imaginary on the surface and the reflection of the surface in the dock, while the left hand sides are the operators which correspond to the boundary conditions on the surface and the reflection of the surface in the dock.

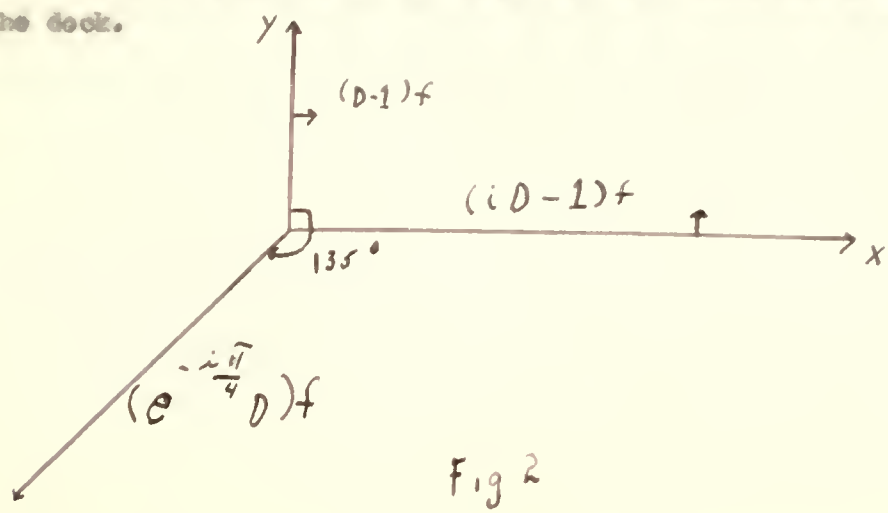

Fig 2

The solutions f_0, f_1, f_{-1}* which satisfy the boundary conditions 3) and 4) are:

8) $$f_0 = e^{i\pi/4}\, e^{z} + e^{-i\pi/4}\, e^{-iz} \quad .$$

9) $$f_1 = e^{i\pi/4}\, e^{z}\int_0^{z} e^{-t}\, t^{\frac{1}{3}}\, dt + e^{-i\pi/4}\, e^{-iz}\int_0^{-iz} e^{-t}\, t^{\frac{1}{3}}\, dt$$

10) $$f_{-1} = e^{i\pi/4}\, e^{z}\int_1^{z} \frac{e^{-t}}{t}\, dt + e^{-i\pi/4}\, e^{-iz}\int_1^{-iz} \frac{e^{-t}}{t}\, dt \, ,$$

where the integrals $\int_0^{-iz} e^{-t}\, t^{\frac{1}{3}}\, dt$ and $\int_1^{-iz} \frac{e^{-t}}{t}\, dt$ are taken over a path which keeps the origin to the left (or $-iz = e^{i\pi/2} z$).

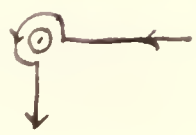

Fig 3

After studying the asymptotic behavior** of the functions f_j, we set

11) $$F_0 = \frac{1}{\sqrt{3}} \left[f_0 - \frac{f_1}{\Gamma\left(\frac{4}{3}\right)} \right]$$

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* See Appendix - Part A

** See Appendix - Part B

12) $$F_1 = \frac{1}{\pi}\left[f_{-1} - f_0 \int_1^{\infty} \frac{e^{-t}}{t}\, dt\right] + F_0$$

and find that

13) $$F_0 \cong e^{-i\pi \frac{11}{12}} e^{-iz} \quad , \quad F_1 \cong e^{-i\pi \frac{5}{12}} e^{-iz} \quad \text{as } |z| \to \infty .$$

We then obtain Φ_0 and Φ_1 as the real parts of the functions F_0 and F_1. The graphs of the values of Φ_0 and Φ_1 on $y = 0$ are shown in Fig. 4.

The asymptotic formulas for Φ_0 and Φ_1 obtained from (13) differ from Φ_0 and Φ_1 by less than 1% for $x \geq 5$.

A short table of the integrals appearing in f_1 together with a description of the method of computation is in Part C of the appendix.

We observe that the ratio

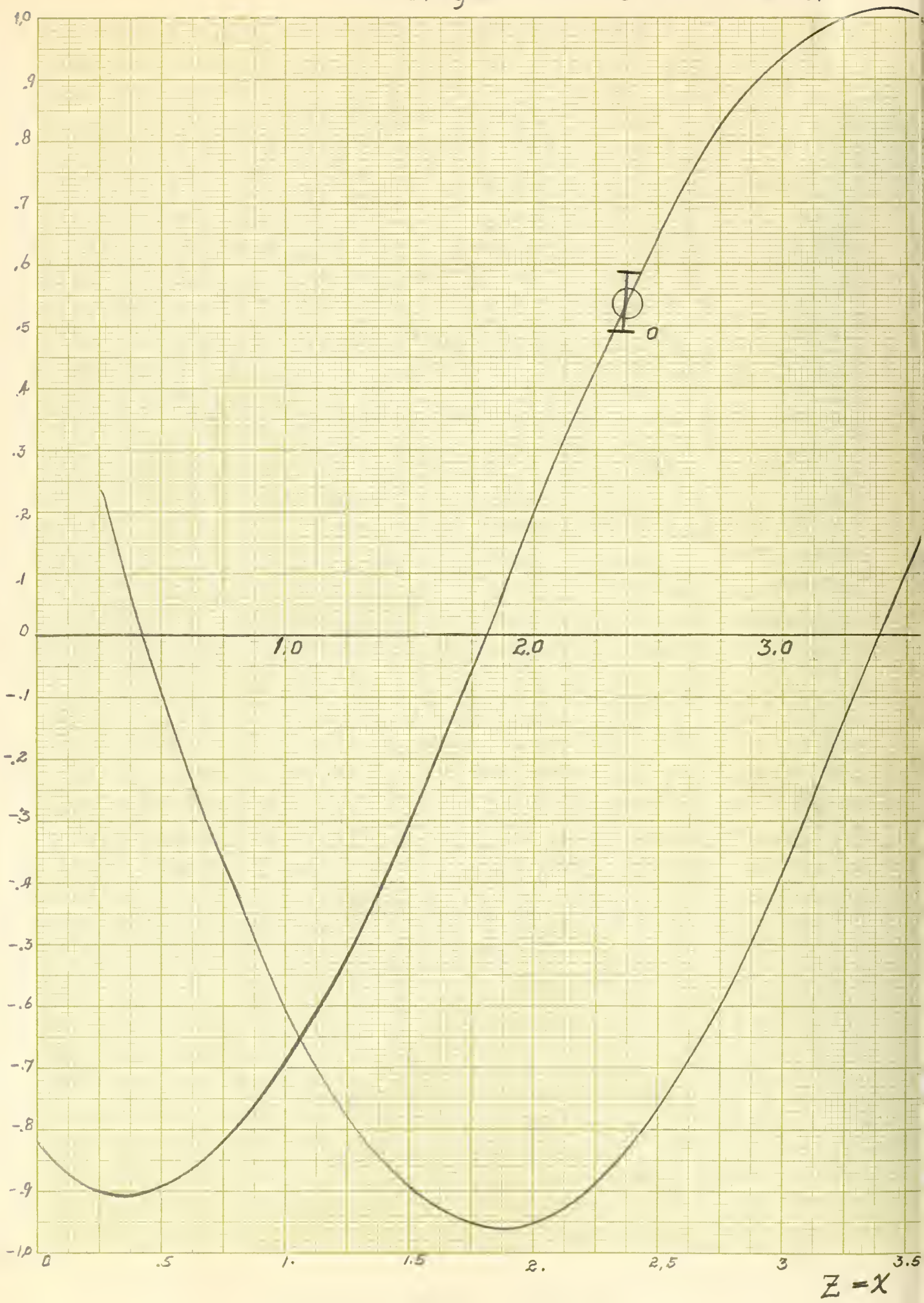
$$\frac{\text{amplitude of } \Phi_0 \text{ at origin}}{\text{amplitude of } \Phi_0 \text{ at infinity}} = \sqrt{\frac{q}{p}} = \sqrt{\frac{2}{3}} .$$

On the other hand, the first maximum of Φ_0 (and Φ_1) is at least 1 percent higher than the maximum amplitude at ∞. Further study of this problem is necessary, probably along the lines of the work of K.O. Friedrichs, in order to obtain an overall picture of the path of the crests of the progressing wave.

The progressing wave is of the form

$$\Phi_0 \sin t + \Phi_1 \cos t \cong \cos(x + \frac{5}{12}\pi + t) \text{, for large } x$$

Standing Wave Solutions on Surface.



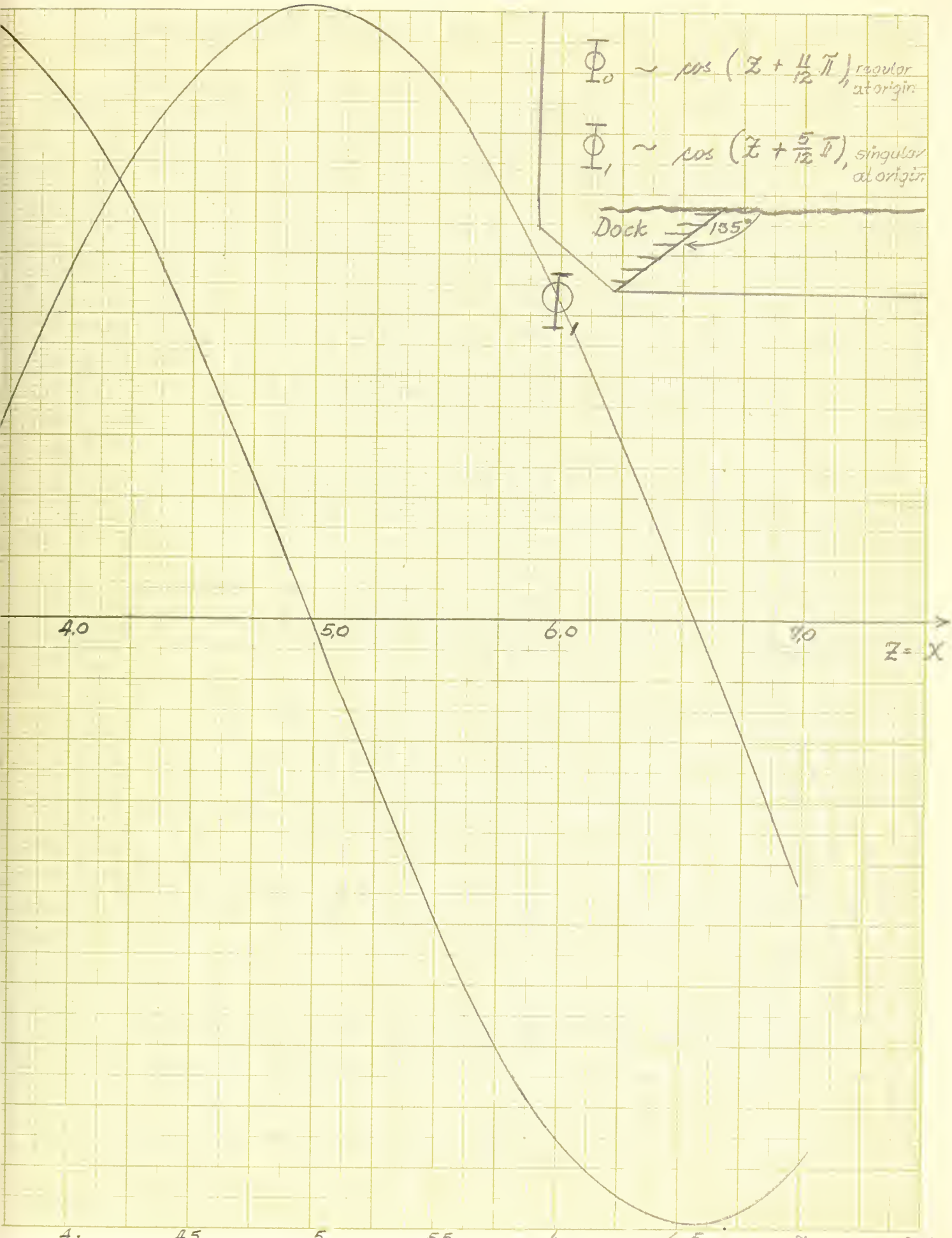

$\Phi_0 \sim \cos\left(Z + \frac{11}{12}\pi\right)$, regular at origin
$\Phi_1 \sim \cos\left(Z + \frac{5}{12}\pi\right)$, singular at origin
Dock
135°
Φ_1
4.0
5.0
6.0
7.0
$Z = X$
4.
4.5
5.
5.5
6
6.5
7.
7.5

Fig 4

APPENDIX

Part A - Boundary Conditions

We now verify that the functions f_j satisfy the boundary conditions

3) $\mathrm{Re}(iD-1)f_j = 0$ for $z = x,\ x > 0$

4) $\mathrm{Re}(e^{-i\frac{\pi}{4}}D)f_j = 0$ for $z = re^{-i\pi 3/4},\ r > 0.$

$3)_0$: $(iD-1)f_0 = (i-1)e^{i\pi 3/4}e^{z}$

$= -i\sqrt{2}\,e^{z}$ which is pure imaginary

for z real.

$4)_0$: $e^{-\frac{i\pi}{4}}D f_0 = e^{\frac{i\pi}{2}}e^{z} - i\,e^{-i\pi}\,e^{-iz},\ z = re^{-i\pi 3/4}$

$= i\left[e^{re^{-i\pi 3/4}} + e^{re^{i\pi 3/4}}\right]$ which is pure imaginary.

$3)_1$: $(iD-1)F_1 = (i-1)e^{i\pi 3/4}\,e^{z}\int_0^z e^{-t}t^{\frac{1}{3}}\,dt + e^{-i\pi 3/4}z^{\frac{1}{3}} + e^{-i\pi 3/4}(-iz)^{\frac{1}{3}}$

$= -i\left(2e^{z}\int_0^z e^{-t}t^{\frac{1}{3}}\,dt + e^{-i\pi 3/4}z^{\frac{1}{3}} + e^{-\frac{i\pi}{4}}z^{\frac{1}{3}}\right)$

which is pure imaginary. It is here that we determine the path of integration ~~to yield~~ so that $-iz = e^{i\pi 3/2}z$.

$$4)_1: \quad e^{-i\pi/4} D f_1 = ie^{z}\int_0^{z} e^{-t} t^{\frac{1}{3}} dt + ie^{-iz}\int_0^{-iz} e^{-t} t^{\frac{1}{3}} + i(-iz)^{\frac{1}{3}} \; ; \qquad z = re^{-i\pi 3/4}$$

$$= i\left[e^{re^{-i\pi 3/4}} \int_0^{re^{-i\pi 3/4}} e^{-t} t^{\frac{1}{3}} dt + e^{re^{i\pi 3/4}} \int_0^{re^{i\pi 3/4}} e^{-t} t^{\frac{1}{3}} dt + \left(e^{-\frac{i\pi}{4}} r^{\frac{1}{3}} + e^{\frac{i\pi}{4}} r^{\frac{1}{3}} \right) \right]$$

which is pure imaginary. (We have again made use of the definition $-iz = e^{i\pi 3/2} z$.)

$$3)_{-1}: \quad (iD-1) f_{-1} = (i-1) e^{i\pi 3/4} e^{z} \int_1^{z} \frac{e^{-t}}{t} dt + \left(\frac{e^{-i\pi 3/4}}{z} + \frac{e^{\frac{-i\pi}{4}}}{z} \right)$$

which is pure imaginary for z real.

$$4)_{-1}: \quad e^{-i\pi/4} D f_{-1} = ie^{z} \int_1^{z} \frac{e^{-t}}{t} dt + ie^{-iz} \int_1^{-iz} \frac{e^{-t}}{t} dt + \frac{i}{z} - \frac{1}{z} \; ; z = re^{-i\pi 3/4}$$

$$= i\left[e^{re^{-i\pi 3/4}} \int_1^{re^{-i\pi 3/4}} \frac{e^{-t}}{t} dt + e^{re^{i\pi 3/4}} \int_1^{re^{i\pi 3/4}} \frac{e^{-t}}{t} dt + \left(\frac{e^{-i\pi 3/4}}{r} - \frac{e^{i\pi 3/4}}{r} \right) \right]$$

which is pure imaginary.

Part B - Behavior at Infinity

There is no trouble with $f_o = e^{i\pi 3/4} e^{z} + e^{-i\pi 3/4} e^{-iz}$. However, we observe that f_o is unbounded at infinity and consequently we would like to have the function f_1 behave in such a way that a linear combination of f_o and f_1 would be bounded at infinity.

We, therefore, expand the integrals contained in f_1 about infinity:

$$\int_{0\rightarrow}^{z} e^{-t}\, t^{\frac{1}{3}}\, dt = \int_{0}^{\infty} e^{-t} t^{\frac{1}{3}} dt - \int_{z}^{\infty} e^{-t} t^{\frac{1}{3}}\, dt\,,$$

so that by integration by parts of the second term

$$14) \qquad \int_{0\rightarrow}^{z} e^{-t} t^{\frac{1}{3}} dt = \Gamma\left(\frac{4}{3}\right) - e^{-z}\left[z^{\frac{1}{3}} + \frac{2}{z^{2/3}} + \cdots\right].$$

On the other hand,

$$\int_{0\,\circlearrowleft}^{-iz} e^{-t}\, t^{\frac{1}{3}} dt = \int_{0}^{\infty} e^{-t} t^{\frac{1}{3}}\, dt - \int_{-iz}^{\infty -iz} e^{-t} t^{\frac{1}{3}}\, dt,$$

where in the first integral on the right hand side of the equation $t = e^{2\pi i} t$ and in the second integral $-iz = e^{i\pi 3/2} z$.

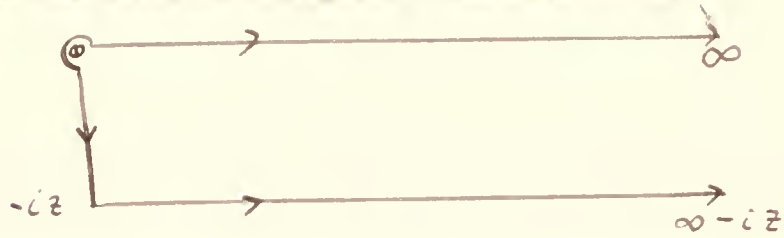


Fig 5

Consequently, integration by parts of the second term yields

15) $$\int_{0}^{-iz} e^{-t}\, t^{\frac{1}{3}} dt = e^{\frac{2\pi i}{3}} \Gamma\left(\tfrac{4}{3}\right) - e^{iz}\left[(-iz)^{\frac{1}{3}} + \frac{b}{(-iz)^{2/3}} + \dots\right]$$

From 14), 15) and 11) we find that $F_0 \cong e^{\frac{-i\pi 11}{12}} e^{-iz}$, which verifies the statement of equation 13).

Next we study the integrals in f_{-1}:

$$\int_{1}^{z} \frac{e^{-t}}{t}\, dt = \int_{1}^{\infty} \frac{e^{-t}}{t}\, dt - \int_{z}^{\infty} \frac{e^{-t}}{t}\, dt$$

and integration by parts of the second integral gives us

16) $$\int_{1}^{z} \frac{e^{-t}}{t}\, dt = \int_{1}^{\infty} \frac{e^{-t}}{t}\, dt - e^{-z}\left[\frac{1}{z} + \frac{c}{z^2} + \dots\right].$$

The remaining integral can be handled as follows:

$$\int_{1}^{-iz} \frac{e^{-t}}{t}\, dt = \int_{1}^{\infty} \frac{e^{-t}}{t}\, dt - \int_{-iz}^{\infty} \frac{e^{-t}}{t}\, dt + 2\pi i.$$

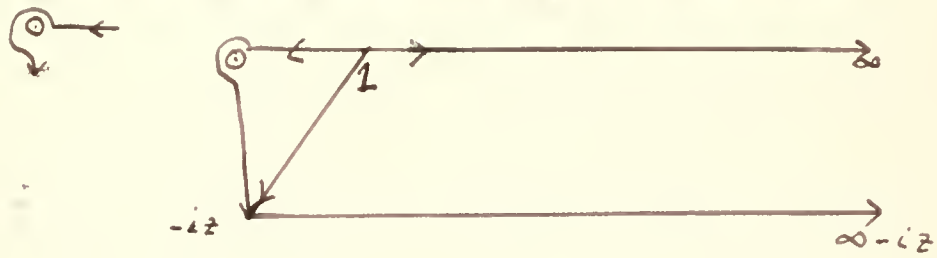


Fig 6

The number $2\pi i$ is added because the integrand has a simple pole at the origin which is circled in the positive direction (see Fig.5). We can now write down the expansion

$$17)\quad \int_{1}^{-iz} \frac{e^{-t}}{t}\,dt = \int_{1}^{\infty} \frac{e^{-t}}{t}\,dt + 2\pi i - e^{iz}\left[\frac{1}{-iz} + \frac{d}{(-iz)^2} + \dots\right]$$

With the results of 14), 15), 16) and 17) it is a simple matter to verify that the definition of F_1 in 12) leads to

$$F_1 \cong e^{-i\pi 5/12}\, e^{-iz}\,.$$

Part C - Table of Exponential Integrals

The integrals appearing in f_1 were evaluated as follows:

(a) for $|z| \leq 1$, we replaced e^{-t} by its power series about the origin, multiplied by $t^{1/3}$ and then integrated term by term.

(b) for $|z| \geq 1$, we evaluated $e^{-t}t^{1/3}$, $(\sin t)t^{1/3}$, and $(\cos t)t^{1/3}$ at intervals of $\Delta t = .25$ and used numerical integration up to $t = 7$.

A short table of values of these integrals correct to within 1 unit in the fourth decimal place may prove useful in similar computations.

z	$\int^{ze^{i\pi 3/2}} e^{-t} t^{1/3} dt$ Real	$\int^{ze^{i\pi 3/2}} e^{-t} t^{1/3} dt$ Imaginary	$\int_0^z e^{-t} t^{1/3} dt$
0.00	0.0000	0.0000	0.0000
.25	+.1166	+.0168	.1026
.50	+.2829	+.0831	.2258
.75	+.4552	+.2078	.3400
1.00	+.6076	+.3914	.4396
1.25	+.7190	+.6255	.5241
1.50	+.7726	+.5974	.5944
1.75	+.7564	+1.1901	.6524
2.00	+.6641	+1.4833	.6997
2.25	+.4952	+1.7557	.7382
2.50	+.2554	+1.9863	.7693
2.75	-.0438	+2.1[illegible]	.7943
3.00	-.3058	+2.2492	.8144
3.25	-.7503	+2.2550	.8305
3.50	-1.1142	+2.1683	.8434
3.75	-1.4532	+1.9901	.8536
4.00	-1.7440	+1.7278	.8618
4.25	-1.9555	+1.3949	.8683
4.50	-2.1003	+1.0100	.8734
4.75	-2.1364	+ .5961	.8775
5.00	-2.0578	+ .1789	.8808
5.25	-1.8953	- .2148	.8833
5.50	-1.6265	- .5591	.8854
5.75	-1.2756	- .8302	.8870
6.00	-.8626	-1.0087	.8882
6.25	-.4121	-1.0804	.8892
6.50	+.0493	-1.0379	.8900
6.75	+.4895	- .8908	.8907
7.00	+.8829	- .6182	.8912

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